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CYCLE-CREATING BIFURCATION FROM A FAMILY OF EQUILIBRIA OF A DYNAMICAL SYSTEM AND ITS DELAY†

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For a dynamical system with cosymmetry, a study is made of the bifurcation in which a cycle branches off from an equilibrium in a continuous one-parameter family of equilibria, as the parameter passes through a critical value. Unlike the classical situation that occurs when the equilibrium is isolated, a self-excited oscillatory mode generally branches off with delay relative to the parameter. Another characteristic difference is the possibility of supercritical branching of an unstable limit cycle.

Cosymmetry in a dynamical system may be a natural reason for the existence of a continuous family of equilibria [1, 2]. In a parametrized system such families may branch off from isolated equilibria. This type of bifurcation occurs in the problem of plane seepage convection of single-component [1, 3] and multicomponent liquids; the presence of heat sources is also allowed.‡ Cosymmetry and the attendant bifurcation phenomena also arise in problems of classical mechanics, for example when the potential energy of a natural mechanical system is invariant with respect to some transformation group.§

It is natural to ask whether further bifurcations of the family may occur as the parameter is varied. In addition to the static bifurcations of collapse, merging, creation and disappearance of families of equilibria, a local bifurcation may also occur due to the onset of oscillatory instability of equilibria in the family and the evolution of a self-excited oscillatory periodic mode of motion. This is the topic of the present paper, which presents the contents of two previous preprints.¶, ‖

There are different approaches to the problem of the onset of periodic self-excited oscillations [4, 5]. In this paper we will use the Lyapunov–Schmidt method, as developed in [6]. Fortunately, when this method is applied to specific problems it is not necessary to return each time to an analysis of the branching equation. One simply looks for a solution as a power series in the supercriticality or neutral amplitude [7].

When a neutral oscillatory mode appears in one of the equilibria of an initially stable family and the parameter is increased further, this generally leads to the appearance of only a small unstable arc. A limit cycle branches off from one end of the arc not immediately, but at a larger parameter value. This effect—a delay of the cycle-creation bifurcation—is a new specific feature of systems with cosymmetry. Also worthy of mention is the possibility of supercritical branching of an unstable cycle. For brevity—and only for that reason—we will confine our attention here to analytic vector fields in a Hilbert space. The passage to abstract parabolic equations in a Banach space and to the equations of mathematical physics is performed directly, without any change in the formalism (see [6]).

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1. FORMULATION OF THE PROBLEM

Consider an ordinary differential equation with a real parameter λ in a Hilbert space H

$$\dot{\theta} = F(\theta, \lambda) \tag{1.1}$$

We will assume that the operator $F\colon H \times \mathbb{R} \to H$ is analytic and that it admits of a cosymmetry $L\colon H \to H$ independent of λ. This means that for all $\theta \in H$, $\lambda \in \mathbb{R}$,

$$(F(\theta, \lambda), L\theta) = 0 \tag{1.2}$$

Suppose that, for some value λ_0 of the parameter λ, there is a known non-cosymmetric solution θ_0 of Eq. (1.1). This means that

$$F(\theta_0, \lambda_0) = 0, \quad L\theta_0 \neq 0 \tag{1.3}$$

Let us assume that the spectrum $\sigma(A_0)$ of the derivative $A_0 = F'_\theta(\theta_0, \lambda_0)$ is a union of three spectral sets $\sigma_+(A_0)$, $\sigma_-(A_0)$, $\sigma_0(A_0)$ situated in the interior of the right half-plane, in the interior of the left half-plane and on the imaginary axis respectively. It will also be assumed that the neutral spectrum $\sigma_0(A_0)$ contains three simple eigenvalues 0, $\pm i\omega_0$ but no other points of the imaginary axis; actually, the only essential condition is that it should contain no points $in\omega_0$ for $n = \pm 2, \pm 3$.

Note that the point 0 belongs to the spectrum $\sigma(A_0)$, because $L\theta_0$ is an eigenvector of the adjoint operator A_0^* [1]

$$A_0^* L\theta_0 = 0 \tag{1.4}$$

Our assumption that zero is a simple eigenvalue means that there is no further degeneracy, apart from the unavoidable degeneracy due to the existence of the cosymmetry (1.2) and the non-cosymmetry of the equilibrium ($L\theta_0 \neq 0$). Under these conditions, the cosymmetric version of the Implicit Function theorem is applicable (Theorem 1 in [1]; for generalizations see [8, 9]). Hence Eq. (1.1) has a one-parameter family of equilibria $s \mapsto c(s)$, $s \in (-\eta, \eta)$ for some $\eta > 0$, so that

$$F(c(s), \lambda_0) = 0, \quad c(0) = \theta_0 \tag{1.5}$$

In this situation there are no other equilibria in the neighbourhood of the point θ_0, and 0 is a simple eigenvalue of the operator $A_s = F'_\theta(c(s), \lambda_0$ for all s. The corresponding eigenvector is $\gamma_s = c'(s)$—a tangent vector to the family. As the eigenvalue is simple, we have $(\gamma_0, L\theta_0) \neq 0$, so that, scaling the parameter s if necessary, we may assume that

$$A_0\gamma_0 = 0, \quad (\gamma_0, L\theta_0) = 1 \tag{1.6}$$

When λ, being varied, passes through λ_0, the family $c(s)$ does not disappear but is only slightly deformed, and in this sense it is regular. This follows from the cosymmetric version of the Implicit Function theorem with a parameter [8, 9], but it will be established below independently.

The fact that the spectrum of A_0 contains pure imaginary eigenvalues means that λ_0 is a critical value of oscillatory instability of the equilibrium θ_0. It is natural to ask whether the equilibria of the family are stable at near-critical values of λ and whether limit cycles may branch off from the family of equilibria $c(s)$. The present paper is devoted to answering these questions.

2. THE LINEARIZED EQUATION

Introducing a new time variable $\tau = \omega_0 t$, we write the equation, linearized at the equilibrium θ_0, as

$$Tu \equiv \omega_0 u' - A_0 u = f \tag{2.1}$$

where the prime denotes differentiation with respect to τ and $f = f(\tau)$ is a given continuous 2π-periodic vector function. We wish to find a 2π-periodic solution of Eq. (2.1).

The homogeneous equation $Tu = 0$ has three independent 2π-periodic solutions: γ_0, $\varphi e^{i\tau}$, $\varphi^* e^{-i\tau}$, where φ is an eigenvector of the operator A_0: $A_0\varphi = i\omega_0\varphi$. The homogeneous adjoint equation

$$T^*w \equiv -\omega_0 w' - A_0^* w = 0 \tag{2.2}$$

also has three independent periodic solutions: $L\theta_0$, $\Phi e^{i\tau}$, $\Phi^* e^{-i\tau}$, where Φ is an eigenvector of the operator A_0^*

$$A_0^*\Phi = -i\omega_0\Phi \tag{2.3}$$

Since we have stipulated that $\pm i\omega_0$ are simple eigenvalues, we can introduce the normalization

$$(\varphi, \Phi) = 1 \tag{2.4}$$

Let E_+, E_0 be the spectral projections corresponding to the spectral sets $\sigma_+(A_0)$, $\sigma_-(A_0)$, $\sigma_0(A_0)$. The projection E_0 is given by the formula

$$E_0 h = (h, L\theta_0)\gamma_0 + (h,\Phi)\varphi + (h,\Phi^*)\varphi^* \tag{2.5}$$

for any h of the complex hull H^c of the space H. We also define the complementary projection E_0' by $E_0' = I - E_0 = E_+ + E_-$.

Lemma (The Solvability Condition). Equation (2.1) with a real continuous vector function f has a 2π-continuous solution if and only if f satisfies the following orthogonality condition

$$\left\langle (f,\Phi)e^{-i\tau}\right\rangle = 0, \quad \left(\langle f\rangle, L\theta_0\right) = 0 \tag{2.6}$$

where the angular brackets denote averaging over time

$$\langle f\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\tau)d\tau \tag{2.7}$$

If condition (2.6) is satisfied, all 2π-periodic solutions of Eq. (2.1) are defined by the equality

$$u(\tau) = \int_{-\infty}^{\tau} e^{(\tau-s)A_0^-}E_- f(s)ds - \int_{\tau}^{\infty} e^{(\tau-s)A_0^+}E_+ f(s)ds + \alpha\varphi e^{i\tau} + \alpha^*\varphi^* e^{-i\tau} + \beta\gamma_0 \tag{2.8}$$

where A_0^-, A_0^+ are the restrictions of A_0 to the subspaces im E_- and im E_+, respectively, and $\alpha \in \mathbb{C}$, $\beta \in \mathbb{R}$ are arbitrary constants.

This result is a special case of well-known theorems. For general equations with bounded operator coefficients, such theorems may be found in [10]; similar abstract parabolic equations are considered in [11].

Equation (2.1) may also be solved by Fourier analysis. If the solvability conditions (2.6) are satisfied and the vector function f has a Fourier expansion

$$f(\tau) = \sum_{n=-\infty}^{\infty} f_n e^{in\tau}, \quad f_{-n} = f_n^* \tag{2.9}$$

then the periodic solution (2.8) may be written in the form

$$u(\tau) = \sum_{\substack{n=-\infty \\ n\neq 0,\pm 1}}^{\infty} \left(in\omega_0 I - A_0\right)^{-1} f_n e^{in\tau} + (i\omega_0 I' - A_0')^{-1} E_0' f_1 e^{i\tau} - (i\omega_0 I' + A_0')^{-1} E_0' f_{-1} e^{-i\tau} -$$

$$-A_0'^{-1} E_0' f_0 + \alpha\varphi e^{i\tau} + \alpha^*\varphi^* e^{-i\tau} + \beta\gamma_0 \tag{2.10}$$

where A_0' is the restriction of A_0 on the complementary space imE_0' to the neutral subspace imE_0 and I' is the identity operator of this subspace.

We shall need the following decomposition of the Hilbert space $H_p = L_2(S^1, H)$ of vector functions on the circle S^1 with values in H. Let H_p^0 be the subspace of H_p of all vector functions of the form

$$u(\tau) = \alpha\varphi e^{i\tau} + \alpha^*\varphi^* e^{-i\tau} + \beta\gamma_0; \quad \alpha \in \mathbb{C}, \quad \beta \in \mathbb{R} \tag{2.11}$$

Define a projection $\Pi: H_p \to H_p^0$ by setting, for any vector function $f \in H_p$

$$(\Pi f)(\tau)=\left\langle (f,\Phi)e^{-i\tau}\right\rangle e^{i\tau}\varphi+\left\langle (f,\Phi^*)e^{i\tau}\right\rangle e^{-i\tau}\varphi^* +\left(\langle f\rangle, L\theta_0\right)\gamma_0 \tag{2.12}$$

Let H_p' denote the complementary subspace and Π' the complementary projection, so that $H_p = H_p^0 \oplus H_p'$, Π', $\Pi' = I - \Pi$.

3. THE CYCLE-BRANCHING EQUATION

We will seek a periodic solution of Eq. (1.1) with unknown period $p = 2\pi/\omega$, in the form

$$\theta(t)=\theta_0+u(\tau),\quad \tau=\omega t \tag{3.1}$$

Substituting into (1.1), we get an equation for u, ω

$$\omega\frac{du}{d\tau}=F(\theta_0+u,\lambda) \tag{3.2}$$

We define $\lambda = \lambda_0 + \delta$, $\omega = \omega_0 + \mu$, assuming that δ and μ, like u, are small. Equation (3.2) can be rewritten as

$$Tu=-\mu u'+F(\theta_0+u,\lambda_0+\delta)-A_0u \tag{3.3}$$

We now use the Lyapunov–Schmidt method as developed in [6].

Represent u in the form $u = \Pi u + \Pi' u$, or

$$u(\tau)=\alpha\varphi e^{i\tau}+\alpha^*\varphi^* e^{-i\tau}+\beta\gamma_0+v(\tau) \tag{3.4}$$

The vector $v = \Pi' u$ must satisfy the orthogonality conditions

$$\left\langle (v,\Phi)e^{-i\tau}\right\rangle=0,\quad \left(\langle v\rangle, L\theta_0\right)=0 \tag{3.5}$$

where $\alpha \in \mathbb{C}$, $\beta \in \mathbb{R}$ are unknown constants and $v \in H_p'$ is an unknown vector function.

The problem is invariant to time shifts $\tau \to \tau + \tau_0$. Hence, together with any solution α, β, $v(\tau)$, the whole orbit of the action of the rotation group of the circle $\{\alpha e^{i\tau_0}, \beta, v(\tau + \tau_0)\}$ consists of solutions. Choosing τ_0 so that $\alpha > 0$, we rewrite (3.4) as

$$u(\tau)=\alpha\psi(\tau)+\beta\gamma_0+v(\tau),\quad \psi(\tau)=\varphi e^{i\tau}+\varphi^* e^{-i\tau} \tag{3.6}$$

Equation (3.3) is equivalent to the system

$$Tv=\Pi' f \tag{3.7}$$

$$\Pi f=0$$

$$f=-\mu\alpha\psi'-\mu v'+F(\theta_0+\alpha\psi+\beta\gamma_0+v,\lambda_0+\delta)-A_0(\alpha\psi+v) \tag{3.8}$$

Equation (3.8) is equivalent to two equations

$$g\equiv\left\langle (f,\Phi)e^{-i\tau}\right\rangle=0,\quad h\equiv\left(\langle f\rangle, L\theta_0\right)=0 \tag{3.9}$$

Using Eq. (3.7), we express v in terms of α, β, μ, δ. Substitution into (3.9) then yields the system of cycle-branching equations.

The restriction T_1 of the operator T to H_p' is an invertible operator. Hence the Implicit Function theorem is applicable to Eq. (3.7) in a neighbourhood of the point $v = 0$, $\alpha = \beta = \mu = \delta = 0$. For any small α, β, μ and δ, Eq. (3.7) has a unique solution in the neighbourhood of the zero of H_p' which is analytic as a function of α, β, μ and δ

$$v=\sum_{k,l,m,n=0}^{\infty} v_{klmn}\alpha^k\beta^l\mu^m\delta^n \tag{3.10}$$

The coefficients v_{klmn} are determined successively after substituting (3.10) into (3.7). When that is done

one uses the Taylor expansion of the operator F in the neighbourhood of the point θ_0, λ_0 in powers of $\theta-\theta_0=u$, $\lambda-\lambda_0=\delta$

$$F(\theta_0+u,\lambda_0+\delta)=\sum_{k,l=0}^{\infty}\frac{1}{k!l!}F_{\theta^k\lambda^l}(\theta_0,\lambda_0)u^k\delta^l \tag{3.11}$$

The first few terms of this expansion are

$$F(\theta_0+u,\lambda_0+\delta)=A_0u+F_{0\lambda}\delta+\frac{1}{2}F_0''u^2+\delta F_{0\lambda}'u+\frac{\delta^2}{2}F_{0\lambda\lambda}+\frac{1}{6}F_0'''u^3+\frac{\delta}{2}F_{0\lambda}'''u^2+\ldots \tag{3.12}$$

The prime denotes differentiation with respect to θ; the subscript zero indicates that the derivative is to be calculated at the point θ_0, λ_0.

It is convenient to continue in the following order. Expanding F in powers of α, β, μ and δ, we successively examine terms of the first, second, etc. degree. Once a coefficient f_{klmn} has been determined in the course of this recurrent procedure, we can find υ_{klmn} by solving the equation $T\upsilon_{klmn}=\Pi' f_{klmn}$. We then determine the contributions of this term to the branching equation (3.9), which we rewrite as

$$\sum_0^{\infty}g_{klmn}\alpha^k\beta^l\mu^m\delta^n=0,\quad \sum_0^{\infty}h_{klmn}\alpha^k\beta^l\mu^m\delta^n=0 \tag{3.13}$$

The coefficients υ_{klmn}, g_{klmn}, h_{klmn} are given by the equalities

$$\begin{gathered}\upsilon_{klmn}=T_1^{-1}\Pi' f_{klmn},\quad g_{klmn}=\left\langle (f_{klmn},\Phi)e^{-it}\right\rangle\\ h_{klmn}=\left(\left\langle f_{klmn}\right\rangle, L\theta_0\right)\end{gathered} \tag{3.14}$$

Of the first-order terms in the expansion of f, only $f_{0001}=F_{0\lambda}$ does not vanish. The expansions in system (3.13) begin with the second degree, since $F_{0\lambda}$ contains only non-vanishing harmonics, and differentiation of (1.2) gives the relationships

$$(F_{0\lambda},L\theta_0)=(F_{0\lambda\lambda},L\theta_0)=\cdots=(F_{0\lambda^k},L\theta_0)=\cdots=0 \tag{3.15}$$

The expansion for υ is

$$\upsilon=\delta\upsilon_{0001}+\alpha^2\upsilon_{2000}+\alpha\beta\upsilon_{1100}+\alpha\delta\upsilon_{1001}+\beta^2\upsilon_{0200}+\beta\delta\upsilon_{0101}+\delta^2\upsilon_{0002}+\alpha^3\upsilon_{3000}+\ldots \tag{3.16}$$

The coefficients are given by the formulae

$$\begin{gathered}\upsilon_{0001}=T_1^{-1}f_{0001}=-A_0'^{-1}F_{0\lambda},\quad \upsilon_{2000}=\frac{1}{2}T_1^{-1}\Pi' F_0''\psi^2\\ \upsilon_{1100}=T_1^{-1}\Pi' F_0''(\psi,\gamma_0),\quad \upsilon_{1001}=T_1^{-1}\Pi'\left[F_0''(\psi,\upsilon_{0001})+F_{0\lambda}'\psi\right]\\ \upsilon_{0200}=\frac{1}{2}T_1^{-1}\Pi' F_0''\gamma_0^2,\quad \upsilon_{0101}=T_1^{-1}\Pi'\left[F_0''(\gamma_0,\upsilon_{0001})+F_{0\lambda}'\gamma_0\right]\\ \upsilon_{0002}=T_1^{-1}\Pi'\left(\frac{1}{2}F_0''\upsilon_{0001}^2+F_{0\lambda}'\upsilon_{0001}+\frac{1}{2}F_{0\lambda\lambda}\right)\\ \upsilon_{3000}=T_1^{-1}\Pi'\left[\frac{1}{6}F_0'''\psi^3+F_0''(\psi,\upsilon_{2000})\right]\end{gathered} \tag{3.17}$$

Correspondingly the first equation of (3.13) takes the form

$$-i\alpha\mu(\varphi,\Phi)+\alpha\beta g_{1100}+\alpha\delta g_{1001}+\alpha^3 g_{3000}+\cdots=0 \tag{3.18}$$

where we have written down all terms of the expansion up to and including the second degree, retaining only those cubic terms necessary for what follows. We have the following formulae for the coefficients

$$g_{1100} = (F_0''(\varphi, \gamma_0), \Phi), \quad g_{1001} = (F_0''(\varphi, \upsilon_{0001}) + F_{0\lambda}'\varphi, \Phi) \tag{3.19}$$

$$g_{3000} = \left\langle \left(F_0''(\varphi, \upsilon_{2000}), \Phi\right) e^{-i\tau} \right\rangle + \frac{1}{6}\left\langle \left(F_0'''\psi^3, \Phi\right) e^{-i\tau} \right\rangle$$

The expansion of the left-hand side of the second equation of (3.13) may be written in the form

$$\alpha^2 h_{2000} + \alpha^2 \beta h_{2100} + \alpha^2 \delta h_{2001} + \alpha^4 h_{4000} + \cdots = 0 \tag{3.20}$$

$$h_{2000} = \frac{1}{2}\left(\left\langle F_0''\psi^2 \right\rangle, L\theta_0\right) = \left(F_0''(\varphi, \varphi^*), L\theta_0\right), \quad h_{3000} = 0$$

$$h_{2100} = \left(\left\langle F_0''(\psi, \upsilon_{1100})\right\rangle, L\theta_0\right) + \left(F_0''(\gamma_0, \langle \upsilon_{2000}\rangle), L\theta_0\right) + \frac{1}{2}\left(\left\langle F_0'''(\psi, \psi, \gamma_0)\right\rangle, L\theta_0\right)$$

$$h_{2001} = \Big(\left\langle F_0''(\psi, \upsilon_{1001})\right\rangle + \left\langle F_0''(\upsilon_{0001}, \upsilon_{2000})\right\rangle + \left\langle F_{0\lambda}'\upsilon_{2000}\right\rangle + \frac{1}{2}\left\langle F_0'''(\upsilon_{0001}, \psi, \psi)\right\rangle + \frac{1}{2}\left\langle F_{0\lambda}''\psi^2\right\rangle, L\theta_0\Big)$$

$$h_{4000} = \left(\left\langle F_0''(\psi, \upsilon_{3000})\right\rangle + \frac{1}{2}\left\langle F_0''\upsilon_{2000}^2\right\rangle + \frac{1}{2}\left\langle F_0'''(\psi, \psi, \upsilon_{2000})\right\rangle + \frac{1}{24}\left\langle F_0^{\mathrm{IV}}\psi^4\right\rangle, L\theta_0\right)$$

We have taken into consideration here that $\langle \psi \rangle = 0$ and therefore also $\langle \upsilon_{100} \rangle = 0$. Terms in (3.20) with monomials β^m and δ^n also vanish, as do all terms not containing α. This follows from the equality

$$h(0, \beta, \mu, \delta) = 0 \tag{3.21}$$

To prove this, consider Eq. (3.7) for $\alpha = 0$ in equilibrium. The assumptions of the Implicit Function Theorem in H_p' are satisfied as before. By uniqueness, υ depends in this case only on β and δ and not on μ. The second branching equation (3.13) in the problem of equilibria is satisfied identically. Indeed, the equilibrium equation $F(\theta, \lambda) = 0$ is equivalent in the neighbourhood of θ_0, λ_0 to

$$F(\theta, \lambda) - (F(\theta, \lambda), L\theta_0)\gamma_0 = 0 \tag{3.22}$$

This equation obviously holds for equilibria. Conversely, if Eq. (3.22) holds, then, multiplying it scalarly by $L\theta$ and using (1.2) and (1.6), we conclude that the second term in (3.22) vanishes. This takes into account that $(\gamma_0, L\theta) \neq 0$ for θ close to θ_0 – for $\theta = \theta_0$ this quantity equals unity (by (1.6)). We have thus proved that the second branching equation, which may be written in the form $(F(\theta, \lambda), L\theta_0) = 0$, holds identically for equilibria, which proves (3.21). At the same time, we have shown that the family of equilibria $c(s)$, which exist at $\lambda = \lambda_0$, admits of an extension with respect to the parameter λ—this also follows from the general results of [8, 9]. We also note another conclusion from the previous argument: the expansion (3.10) for υ does not contain monomials $B^l\mu^m\delta^n$ with $m > 0$.

It remains to remark that Eq. (3.20) contains monomials $\alpha^k\beta^l\mu^m\delta^n$ with even k only, since a time shift $\tau \mapsto \tau + \pi$ leaves the equation unchanged because of the invariance of time averages to shifts, while, on the other hand, this time shift is equivalent to the change of variable $\alpha \mapsto -\alpha$.

4. CYCLE-CREATING BIFURCATION

Theorem 1. Let the right-hand side F of Eq. (1.1) be analytic in a neighbourhood of the point θ_0, λ_0, where θ_0 is an equilibrium: $F(\theta_0, \lambda_0) = 0$, and let the equation have an analytic cosymmetry independent of the parameter λ. Assume that the following conditions hold:

1. the equilibrium θ_0 is not cosymmetric: $L\theta_0 \neq 0$;

2. the spectrum of the derivate $A_0 = F_\theta'(\theta_0, \lambda_0)$ is the union of a pair of spectral sets σ_-, σ_+ in the interiors of the left and right half-planes, respectively, and a spectral set σ_0 on the imaginary axis containing three simple eigenvalues 0, $\pm i\omega_0$, $\omega_0 > 0$ but not containing any of the points $in\omega_0$, $n = \pm 2, \ldots$.

Then the following statements hold:

1. for small $\delta = \lambda - \lambda_0$ a set of equilibria $c(s, \delta)$ exists which depends analytically on $(s, \delta) \in (-s_0, s_0) \times (-\delta_0, \delta_0)$, $s_0 > 0$, $\delta_0 > 0$, so that $F(c(s, \delta), \lambda_0 + \delta) = 0$;

2. if

$$h_{2000} = (F_0''(\varphi, \varphi^*), L\theta_0) \neq 0 \tag{4.1}$$

then for small δ there are no small limit cycles in the neighbourhood of θ_0.

Proof. It has already been established that in the equilibrium problem β and δ remain arbitrary, but υ is uniquely defined by Eq. (3.7). Consequently, for any small δ and β we find equilibria

$$\theta := c(\beta,\delta) = \theta_0 + \beta\gamma_0 + \delta\upsilon_{0001} + \beta^2\upsilon_{0200} + \beta\delta\upsilon_{0101} + \delta^2\upsilon_{0002} + \dots \tag{4.2}$$

The coefficients are given by formulae (3.17).

The second statement follows at once from Eq. (3.20). This completes the proof.

Theorem 2. In addition to the general assumptions of Theorem 1, assume that $h_{2000} = 0$ and that

$$\Delta = \begin{vmatrix} g^r_{3000} & g^r_{1100} \\ h_{4000} & h_{2100} \end{vmatrix} \neq 0, \quad \Delta_1 = \begin{vmatrix} g^r_{1100} & g^r_{1001} \\ h_{2100} & h_{2001} \end{vmatrix} \neq 0 \tag{4.3}$$

where the superscript r denotes the real part. Then a small limit cycle exists that shrinks as $\lambda \to \lambda_0$ to the equilibrium θ; moreover, this cycle is unique, depends analytically on the parameter $\surd\delta = \surd(\lambda - \lambda_0)$ and can be found from the formula

$$\theta(t) = \theta_0 + \alpha_1\varepsilon\psi(\omega_0 t) + \left[\operatorname{sign}\left(\frac{\Delta_1}{\Delta}\right)(\beta_1\gamma_0 + \upsilon_{0001}) + \alpha_1^2\upsilon_{2000}\right]\varepsilon^2 + \dots \tag{4.4}$$

where terms of order ε^3 and higher have been omitted. The small parameter ε and the amplitude coefficients α_1 and β_1 are given by the equalities

$$\varepsilon = \sqrt{\left(\operatorname{sign}\frac{\Delta_1}{\Delta}\right)\delta}, \quad \alpha_1 = \sqrt{\left|\frac{\Delta_1}{\Delta}\right|}, \quad \beta_1 = \frac{\delta_1}{\Delta}, \quad \delta_1 = \begin{vmatrix} g^r_{1001} & g^r_{3000} \\ h_{2001} & h_{4000} \end{vmatrix} \tag{4.5}$$

This solution is real for $\delta > 0$ if $\Delta_1/\Delta > 0$, and for small $\delta < 0$ if $\Delta_1/\Delta < 0$. The frequency ω of the periodic motion depends on δ and has the form

$$\omega = \omega_0 + \mu_1\delta + \dots, \quad \mu_1 = g^i_{1001} + \left(g^i_{1100}\delta_1 + g^i_{3000}\Delta_1\right)/\Delta \tag{4.6}$$

Proof. Given that $h_{2000} = 0$, dividing the branching equations (3.18) and (3.20) by α and α^2, respectively, we can reduce them to the following form (the superscript i denotes the imaginary part)

$$\begin{gathered} -\mu(\varphi,\Phi) + \alpha^2 g^i_{3000} + \beta g^i_{1100} + \delta g^i_{1001} + \dots = 0 \\ \alpha^2 g^r_{3000} + \beta g^r_{1100} + \delta g^r_{1001} + \dots = 0 \\ \alpha^2 h_{4000} + \beta h_{2100} + \delta h_{2001} + \dots = 0 \end{gathered} \tag{4.7}$$

The left-hand sides of these equations are analytic in α^2, β, δ, μ; we have omitted here terms of the second and higher orders with respect to these variables. The required results now follow directly by application of the Implicit Function theorem.

A simpler expression may be obtained from the coefficient h_{2000}

$$h_{2000} = -2\operatorname{Re}(F_0'\varphi, L_0'\varphi) = 2\omega_0 \operatorname{Im}(\varphi, L_0'\varphi) \tag{4.8}$$

This is readily proved using the equality

$$\left[F_0''(u,\upsilon), L\theta_0\right] + \left[F_0'u, L_0'\upsilon\right] + \left[F_0'\upsilon, L_0'u\right] = 0 \tag{4.9}$$

which holds for any complex vectors u and υ; [,] denotes the bilinear product such that $[u, \upsilon] = (u, \upsilon^*)$.

In order to derive (4.9) it is sufficient to differentiate (1.2) twice with respect to θ, complexify and set $\theta = \theta_0$, $\lambda = \lambda_0$.

One can also obtain the existence condition $h_{2000} = 0$ for small limit cycles using the following equality, which follows from (1.1) and (1.2)

$$(\dot{\theta}, L\theta) = 0 \tag{4.10}$$

After substituting $\theta = \theta_0 + \alpha\psi + \beta\gamma + \upsilon$ (see (3.6)) and averaging over time, one obtains the equality $\langle(\dot{\psi}, L_0'\psi)\rangle = 0$ in the principal order, which, by (4.8), is precisely the condition $h_{2000} = 0$.

Holonomic cosymmetries and halos. A cosymmetry $L: H \to H$ of an operator $F: H \to H$ is said to be holonomic if it is potential, so that $L\theta = \text{grad}\ \varphi(\theta)$. In that case the functional (potential) φ is an integral of the differential equation $\dot{\theta} = F\theta$. A cosymmetry L is said to be quasi-holonomic if it becomes holonomic when multiplied by some nowhere-vanishing functional h, i.e. $h(\theta)L\theta = \text{grad}\ \varphi(\theta)$.

We shall say that a cosymmetry L is essentially non-holonomic if it is not holonomic and not even quasi-holonomic. A cosymmetry may be regarded as a differential 1-form, and then an essentially non-holonomic cosymmetry is simple a contact form [12, 13]. The only holonomic linear operators are the self-adjoint operators, while the skew-symmetric operators are, on the contrary, the "most non-holonomic".

In the problem under consideration, the case of holonomic (quasi-holonomic) cosymmetry is exceptional and easily analysed.

Suppose that the general assumptions of Theorem 1 are satisfied and let L be a holonomic cosymmetry. Then the phase space H is partitioned into invariant level sets of the potential φ, which intersect the set of equilibria transversely near θ_0 at λ near λ_0. This follows from the condition that the zero eigenvalue of the derivative $A_0 = F'(\theta_0, \lambda_0)$ is simple; this requires, in addition to the kernel being one-dimensional, that $(\gamma_0, L\theta_0) \neq 0$. Under the natural conditions that there be no further degeneracies, a local bifurcation occurs in each of the level sets near θ_0: a cycle branches off from the equilibrium. As the parameter λ passes through its critical values $\lambda_*(s)$ (where $\lambda_*(0) = \lambda_0$), a limit cycle branches off from the equilibrium $c(s, \lambda_*(s) - \lambda_0)$. As a result one obtains a *halo* – an invariant surface formed by limit cycles as they branch off. Subject to non-degeneracy conditions at θ_0 when $\lambda = \lambda_0$, which are maintained locally, either all these cycles are supercritical and stable, or they are all subcritical and unstable.

5. STABILITY OF EQUILIBRIA AND OF A LIMIT CYCLE

If the operator $A_0 = F'_\theta(\theta_0, \lambda_0)$ has at least one eigenvalue in the right half-plane, the equilibria of the family $c(s, \delta)$ and the branching limit cycle are unstable. We will therefore assume that the spectrum $\sigma(A_0)$ contains only the spectral set σ_- and the triple of simple eigenvalues $0, \pm i\omega_0$, $\omega_0 > 0$. Then, the analysis of stability reduces to investigating the eigenvalues generated by disturbances of the triple; note that the formulae cited below for the perturbed eigenvalues remain valid in the general case.

To investigate the stability of an equilibrium $c(s, \delta)$, we consider the spectrum of the operator

$$A(s,\delta) = F'(c(s,\delta), \lambda_0 + \delta) = A_0 + sA_{10} + \delta A_{01} + \dots \tag{5.1}$$

which is an analytic function of s and δ in the neighbourhood of the point (0, 0) in the (s, δ) plane. The family $c(s, \delta)$ has the expansion

$$c(s,\delta) = \theta_0 + c_{10}s + c_{01}\delta + c_{20}s^2 + c_{11}s\delta + c_{02}\delta^2 + \dots \tag{5.2}$$

whose coefficients are found from the equilibrium equation

$$F(c(s,\delta), \lambda_0 + \delta) = 0 \tag{5.3}$$

Substituting (5.2) into (5.3) we obtain a chain of equations

$$A_0 c_{10} = 0, \quad A_0 c_{01} = -F_{0\lambda}$$

$$A_0 c_{20} = -\frac{1}{2} F_0'' c_{10}^2, \quad A_0 c_{11} = -F_0''(c_{10}, c_{01}) - F_{0\lambda}' c_{10}$$

$$A_0 c_{02} = -\frac{1}{2} F_0'' c_{01}^2 - F_{0\lambda}' c_{01} - \frac{1}{2} F_{0\lambda\lambda}$$

Here $c_{10} = c_s'(0, 0) = \gamma_0$ is the tangent vector to the family at the point θ_0 when $\lambda = \lambda_0$. The other equations have the standard form $A_0 u = f$ and differ only in the functions f on their right. The solvability condition $(f, L\theta_0) = 0$ is automatically satisfied due to cosymmetry. The uniqueness of the solution is guaranteed by the condition $(u, L\theta_0) = 0$.

The operators $A_{10}, A_{01}, \ldots$ in (5.1) are defined by the formulae

$$A_{10} u = F_0''(c_{10}, u), \quad A_{01} u = F_0''(c_{01}, u) + F_{0\lambda}' u, \quad u \in H$$

The eigenvalue zero is retained in the spectrum of the operator $A(s, \delta)$ and remains simple. The corresponding eigenvector $c_s'(s, \delta)$ is tangent to the family. The eigenvalue $i\omega_0$ of A_0, being simple, generates a simple eigenvalue

$$\sigma = i\omega_0 + s\sigma_{10} + \delta\sigma_{01} + s^2\sigma_{20} + s\delta\sigma_{11} + \delta^2\sigma_{02} + \ldots$$

The eigenvector ψ is also analytic in s and δ

$$\psi = \varphi + s\psi_{10} + \delta\psi_{01} + \ldots$$

The coefficients are found from the equation $A(s, \delta)\psi = \sigma\psi$; the term φ is the unperturbed eigenvector: $A_0\varphi = i\omega_0\varphi$. To determine $\psi_{01}, \psi_{10}, \ldots$, as well as $\sigma_{10}, \sigma_{01}, \ldots$, we have the equations

$$T_0\psi_{10} = (\sigma_{10} I - A_{10})\varphi, \quad T_0\psi_{01} = (\sigma_{01} I - A_{01})\varphi$$

$$T_0\psi_{20} = (\sigma_{10} I - A_{10})\psi_{10} + (\sigma_{20} I - A_{20})\varphi, \ldots, \quad T_0 = A_0 - i\omega_0 I$$

The conditions for these equations to be solvable yield the equalities

$$\sigma_{10} = (A_{10}\varphi, \Phi), \qquad \sigma_{01} = (A_{01}\varphi, \Phi)$$

$$\sigma_{20} = ((A_{10} - \sigma_{10} I)\psi_{10}, \Phi) + (A_{20}\varphi, \Phi)$$

where Φ is an eigenvector of the operator A_0^*: $A_0^*\Phi = -i\omega_0\Phi$.

The problem of whether the equilibrium $c(s, \delta)$ is stable may be solved by investigating the sign of the quantity

$$\operatorname{Re}\sigma = s\operatorname{Re}\sigma_{10} + \delta\operatorname{Re}\sigma_{01} + \ldots$$

In the general case, $\operatorname{Re}\sigma_{10} \neq 0$, and then the equilibrium θ_0 at $\delta = 0$ is the common boundary of the stable and unstable arcs of the equilibrium family. By the Implicit Function theorem, applied to the equation $\operatorname{Re}\sigma = 0$, for small δ the family is divided, as before, into a stable arc and an unstable arc by an equilibrium $c(s_0(\delta), \delta)$, where $s_0(\delta)$ is determined from the condition $\operatorname{Re}\sigma = 0$.

An interesting case occurs when λ_0 is the first critical value of oscillatory instability of the equilibria. This means that $\operatorname{Re}\sigma < 0$ for $\delta = 0$ and small $s \neq 0$. In that case $\operatorname{Re}\sigma_{10} = 0$. In the general position, $\operatorname{Re}\sigma_{20} < 0$, $\operatorname{Re}\sigma_{01} \neq 0$, and the principal part of the expansion of $\operatorname{Re}\sigma$ is $\operatorname{Re}\sigma = s^2\operatorname{Re}\sigma_{20} + \delta\operatorname{Re}\sigma_{01} + \ldots$. For small δ such that $\delta\operatorname{Re}\sigma_{01} < 0$, the local family is stable. But if $\delta\operatorname{Re}\sigma_{01} > 0$, a small unstable arc exists whose endpoints $s_\pm = s_\pm(\delta)$ are found from the equation $\operatorname{Re}\sigma = 0$, so that

$$s_\pm = \pm\sqrt{\frac{-\operatorname{Re}\sigma_{01}}{\operatorname{Re}\sigma_{20}}\delta} + O(\delta), \quad \delta \to 0 \tag{5.4}$$

The unstable arc (s_-, s_+) is enclosed by stable arcs. It is clear that the newly formed unstable arc at first grows rapidly.

The results may be summarized as follows.

Proposition 1. Let θ_0 be a non-cosymmetric equilibrium for $\lambda = \lambda_0$, so that $F(\theta_0, \lambda_0) = 0$, $L\theta_0 \neq 0$.

Assume that the spectrum of the operator A_0 satisfies the conditions specified at the beginning of this section. Then the following statements hold.

1. For small s and $\delta = \lambda - \lambda_0$, there is a well-defined family of equilibria $c(s, \delta)$ which depends analytically on s and δ near the point (0, 0).

2. A pure imaginary eigenvalue $i\omega_0$ generates in the stability spectrum of an equilibrium $c(s, \delta)$ an eigenvalue $\sigma = \sigma(s, \delta)$ which depends analytically on s and δ and is such that $\sigma(0, 0) = i\omega_0$.

3. If Re $\sigma_{10} \neq 0$, then for small δ the local family $c(s, \delta)$ contains a unique neutrally stable equilibrium $\theta_0(\delta) = c(s_0(\delta),\delta)$ which depends analytically on δ and divides the family into two arcs—stable and unstable.

4. If Re $\sigma_{10} = 0$, Re $\sigma_{20} \neq 0$ and Re $\sigma_{01} \neq 0$, then as δ passes through zero one obtains on the stable arc at $\delta = 0$ a neutrally stable equilibrium θ_0, which is the first point of an unstable arc which increases rapidly (by root law (5.4)).

It remains to investigate the stability of the limit cycle that branches off. It is generated by a solution θ_c which is $(2\pi/\omega)$-periodic in t and may be written in the form

$$\theta_c = \theta_c(\tau) = \theta_0 + \varepsilon\theta_1(\tau) + \varepsilon^2\theta_2(\tau) + \dots \tag{5.5}$$

$$\omega = \omega_0 + \mu = \omega_0 + \omega_1\delta + \omega_2\delta^2 + \dots$$

Once the existence of the solution and the convergence of the series (5.5) have been proved (see Section 3), the coefficients θ_j and ω_j are most simply found by direct substitution into Eq. (1.1).

The stability of solution (5.5) may be investigated by the linearization method. Seeking a Floquet solution $\exp(\sigma t)u(t)$, where u is a $(2\pi/\omega)$-periodic vector function of t, we arrive at an eigenvalue problem for (2π)-periodic vector functions of τ

$$\omega\frac{du}{d\tau} + \sigma u = A(\tau)u \tag{5.6}$$

$$A(\tau) = F'(\theta_c(\tau), \lambda_0 + \varepsilon^2) = A_0 + \varepsilon A_1(\tau) + \varepsilon^2 A_2(\tau) + \dots$$

This eigenvalue problem can be treated by the methods of analytical perturbation theory, as is done in [6]. It then suffices to study a perturbation of the eigenvalues (Floquet exponents) 0, $\pm i\omega_0$. However, an answer is obtained more rapidly—at least, in the case of the general position—if one notes, first, that instability is possible only on the neutral manifold of the equilibrium θ_0 (in the extended system, where the parameter ε is added as a new variable), and, second, that the Lyapunov–Schmidt method furnishes the solution specifically on the neutral manifold. Hence, analysis of the three-dimensional equation on the neutral manifold yields the same formulae as perturbation theory.

To calculate the leading term of the equation on the neutral manifold, one can also use the method of averaging, which produces the following asymptotic formula for the solution of the Cauchy problem (and for the periodic solution) (see [4, 5])

$$\theta_c(t) = \theta_0 + \varepsilon\alpha_c(t')e^{i\omega_0 t}\varphi + \varepsilon\alpha_c^*(t')e^{-i\omega_0 t}\varphi^* + \varepsilon^2\beta(t')\gamma_0 + \dots$$

where $t' = \varepsilon^2 t$ is slow time. The equations for the unknown amplitudes α_c and β are immediately obtained in normal form by averaging over fast time t (or by the Lyapunov–Schmidt method)

$$\dot{\alpha}_c = \alpha_c\left(g_{3000}|\alpha_c|^2 + g_{1100}\beta + g_{1001}\delta\right)$$

$$\dot{\beta} = |\alpha_c|^2\left(h_{4000}|\alpha_c|^2 + h_{2100}\beta + h_{2001}\delta\right)$$

Changing in the plane of the variable α_c to polar coordinates $\alpha = |\alpha_c|$ and arg α_c, we separate out a system for α and β, which may be written in the form

$$\dot{\alpha} = \alpha(a\alpha^2 + b\beta - p\delta), \quad \dot{\beta} = \alpha^2(c\alpha^2 + d\beta - q\delta)$$

where we have used the notation $g_{3000}^r = a$, $g_{1100}^r = b$, $h_{4000} = c$, $h_{2100} = d$, $g_{1001}^r = -p$, $h_{2001} = -q$. This system has a family of equilibria for which $\alpha = 0$, $\beta = \beta_0$ is arbitrary. For equilibria with zero α one has a system

$$a\alpha^2 + b\beta = p\delta, \quad c\alpha^2 + d\beta = q\delta$$

The solution α_δ, β_δ of this system is precisely the principal approximation of the Lyapunov–Schmidt method for the amplitude of the cycle

$$\alpha_\delta = \sqrt{\frac{\Delta_1}{\Delta}\delta}, \quad \beta_\delta = \frac{\delta_1}{\Delta}\delta$$

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} p & b \\ q & d \end{vmatrix}, \quad \delta_1 = \begin{vmatrix} a & p \\ c & q \end{vmatrix}$$

Using linearization to investigate the stability of the equilibria of the family $\alpha = 0$, $\beta = \beta_0$ and the equilibrium α_δ, β_δ corresponding to the limit cycle, we obtain the following results.

Proposition 2. Under the assumptions of Theorem 2, a limit cycle branches off in the domain $\delta > 0$ ($\delta < 0$) if $\Delta\Delta_1 > 0$ ($\Delta\Delta_1 < 0$). It is stable if the two inequalities $2a + d < 0$, $\Delta > 0$ hold simultaneously, and unstable if at least one of them is badly violated: $\max\{2a + d, -\Delta\} > 0$.

Now let λ_0 be the critical value of λ corresponding to the first loss of stability of the equilibria of the family, so that

$$b = \operatorname{Re}\sigma_{10} = 0, \quad \operatorname{Re}\sigma_{20} < 0, \quad \operatorname{Re}\sigma_{01} = -p \neq 0 \tag{5.7}$$

To fix our ideas, we assume that $p < 0$; then a small unstable arc exists for $\delta > 0$.

Proposition 3. Suppose that the assumptions of Theorem 2 and conditions (5.7) are satisfied. Then a limit cycle branches off into the supercritical domain $\delta > 0$ and is stable if $a < 0$ and $d < 0$; it is unstable if $a < 0$ and $d > 0$. A subcritical cycle ($\delta < 0$) exists for $a > 0$ and is unstable.

These two propositions, together with Theorems 1 and 2, explain the effect of delay observed in cycle-creating bifurcation in cosymmetric systems. At the instant the unstable arc of equilibria is formed, the condition of Theorem 2 for the creation of a limit cycle is satisfied only in exceptional cases. The critical value for a limit cycle branching off from an end of the unstable arc is precisely the necessary condition for the condition of Theorem 2 to be satisfied.

Cosymmetry and the delay effect have already been observed in the simplest three-dimensional model of seepage convection with heat sources (see the footnotes † and ‖ on the first page of this article).

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